Conditional inference with a complex sampling: exact computations and Monte Carlo estimations

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Abstract

In survey statistics, the usual technique for estimating a population total consists in summing appropriately weighted variable values for the units in the sample. Different weighting systems exit: sampling weights, GREG weights or calibration weights for example.
In this article, we propose to use the inverse of conditional inclusion probabilities as weighting system. We study examples where an auxiliary information enables to perform an a posteriori stratification of the population. We show that, in these cases, exact computations of the conditional weights are possible.
When the auxiliary information consists in the knowledge of a quantitative variable for all the units of the population, then we show that the conditional weights can be estimated via Monte-Carlo simulations. This method is applied to outlier and strata-Jumper adjustments.

Keywords: Auxiliary information; Conditional inference; Finite population; Inclusion probabilities; Monte Carlo methods; Sampling weights

1 Introduction

The purpose of this article is to give a systematic use of the auxiliary information at the estimation phase by the means of Monte Carlo methods, in a design based approach.

In survey sampling, we often face a situation where we use information about the population (auxiliary information) available only at the estimation phase. For example, this information can be provided by an administration file available only posterior to the collection stage. Another example would be the number of respondents to a survey. It is classical to deal with the non-response mechanism by a second sampling phase (often Poisson sampling conditional to the size of the sample). The size of the respondents sample is known only after the collection.

This information can be compared to its counterpart estimated by the means of the sample. A significant difference typically reveals an unbalanced sample. In order to take this discrepancy into account, it is necessary to re-evaluate our estimations. In practice, two main technics exist: the model-assisted approach (ratio estimator, post-stratification estimator, regression estimator) and the calibration approach. The conditional approach we will develop in this article has been so far mainly a theoretical concept because it involves rather complex computations of the inclusion probabilities. The use of Monte-Carlo methods could be a novelty that would enable the use of conditional approach in practice. In particular, it seems to be very helpful for the treatment of outliers and strata jumpers.

Conditional inference in survey sampling means that, at the estimation phase, the sample selection is modelized by means of a conditional probability. Hence, expectation and variance of the estimators are computed according to this conditional sampling probability. Moreover, we are thus provided with conditional sampling weights with better properties than the original sampling weights, in the sense that they lead to a better balanced sample (or calibrated sample).

Conditional inference is not a new topic and several authors have studied the conditional expectation and variance of estimators, among them: Rao (1985), Robinson(1987), Tillé (1998, 1999) and Andersson (2004). Moreover, one can see that the problematic of conditional inference is close to inference in the context of rejective sampling design. The difference is that in rejective sampling, the conditioning event is controlled by the design, whereas, in conditional inference, the realization of the event is observed.

In section 2, the classical framework of finite population sampling and some notations are presented.

In section 3, we discuss the well-known setting of simple random sampling where we condition on the sizes of the sub-samples on strata (a posteriori stratification). This leads to an alternative estimator to the classical HT estimator. While a large part of the literature deals with the notion of correction of conditional bias, we will directly use the concept of conditional HT estimator (Tillé, 1998), which seems more natural under conditional inference. A simulation study will be performed in order to compare the accuracy of the conditional strategy to the traditional one.

In section 4, the sampling design is a Poisson sampling conditional to sample size n (also called conditional Poisson sampling of size n). We use again the information about the sub-samples sizes to condition on. We show that the conditional probability corresponds exactly to a stratified conditional Poisson sampling and we give recursive formula that enables the calculation of the conditional inclusion probabilities. These results are new.

In section 5, we use a new conditioning statistic. Following Tillé (1998, 1999), we use the non-conditional HT estimation of the mean of the auxiliary variable to condition on. Whereas Tillé uses asymptotical arguments in order to approximate the conditional inclusion probabilities, we prefer to perform Monte Carlo simulations to address a non-asymptotic setting. Note that this idea of using independent replications of the sampling scheme in order to estimate inclusion probabilities when the sampling design is complex has been already proposed by Fattorini (2006) and Thompson and Wu (2008).

In section 6, we apply this method to practical examples: outlier and strata jumper in business survey. This new method to deal with outliers gives good results.

2 The context

Let U be a finite population of size N. The statistical units of the population are indexed by a label $k \in \{1, ..., N\}$. A random sample without replacement s is selected using a probability (sampling design) $p(.)$. $\mathcal{S}$ is the set of the possible samples s. $I_{[k \in s]}$ is the indicator variable which is equal to one when the unit k is in the sample and 0 otherwise. The size of the sample is $n(s) = |s|$. Let $B_k = \{s \in \mathcal{S}, k \in s\} = \{s \in \mathcal{S}, I_{[k \in s]} = 1\}$ be the set of samples that contain k. For a fixed individual k, let $\pi_k = p(B_k)$ be the inclusion probability and let $d_k = \frac{1}{\pi_k}$ be its sampling weight. For any variable z that takes the value z_k on the U-unit k, the sum $t_z = \sum_{k \in U} z_k$ is referred to as the total of z over U. $\widehat{t}_{z,\pi} = \sum_{k \in s} \frac{1}{\pi_k} z_k$ is the Horvitz-Thompson estimator of the total t_z.

Let x be an auxiliary variable that takes the value x_k for the individual k. The x_k are assumed to be known for all the units of U. Such auxiliary information is often used at the sampling stage in order to improve the sampling design. For example, if the auxiliary variable is a categorical variable then the sampling can be stratified. If the auxiliary variable is quantitative, looking for a balanced sampling on the total of x is a natural idea. These methods reduce the size of the initial set of admissible samples. In the second example, $\mathcal{S}_{balanced} = \{s \in \mathcal{S}, \widehat{t}_{x,\pi} = t_x\}$.
We wish to use auxiliary information *after* the sample selection, that is to take advantage of information such as the number of units sampled in each stratum or the estimation of the total t_x given by the Horvitz-Thompson estimator. Let us take an example where the sample consists in 20 men and 80 women, drawn by a simple random sampling of size $n = 100$ among a total population of $N = 200$ with equal inclusion probabilities $\pi_k = 0.5$. And let us assume that we are given *a posteriori* the additional information that the population has 100 men and 100 women. Then it is hard to maintain anymore that the inclusion probability for both men and women was actually 0.5. It seems more sensible to consider that the men sampled had indeed a inclusion probability of 0.2 and a weight of 5. Conditional inference aims at giving some theoretical support to such feelings.

We use the notation $\mathbf{\Phi}(s)$ for the statistic that will be used in the conditioning. $\mathbf{\Phi}(s)$ is a random vector that takes values in $\mathbb{R}^q$. In fact, $\mathbf{\Phi}(s)$ will often be a discrete random vector which takes values in $\{1, ..., n\}^q$. At each possible subset $\varphi \subset \mathbf{\Phi}(\mathcal{S})$

corresponds an event $A_\varphi = \mathbf{\Phi}^{-1}(\varphi) = \{s \in \mathcal{S}, \mathbf{\Phi}(s) \in \varphi\}$.
For example, if the auxiliary variable x_k is the indicator function of a domain, say $x_k = 1$ if the unit k is a man, then we can choose $\mathbf{\Phi}(s) = \sum_{k \in s} I_{[k \in domain]} = n_{domain}$ the sample size in the domain (number of men in the sample). If the auxiliary variable x_k is a quantitative variable, then we can choose $\mathbf{\Phi}(s) = \sum_{k \in s} \frac{x_k}{\pi_k} = \widehat{t}_{x,\pi}$ the Horvitz-Thompson estimator of the total t_x.

3 A posteriori Simple Random Sampling Stratification

3.1 Classical Inference

In this section, the sampling design is a simple random sampling without replacement(SRS) of fixed size n; $\mathcal{S}_{SRS} = \{s \in \mathcal{S}, n(s) = n\}$; $p(s) = 1/\binom{N}{n}$ and the inclusion probability of each individual k is $\pi_k = n/N$. Let y be the variable of study. y takes the value y_k for the individual k. The y_k are observed for all the units of the sample. The Horvitz-Thompson (HT) estimator of the total $t_y = \sum_{k \in U} y_k$ is $\hat{t}_{y,HT} = \sum_{k \in U} \frac{y_k}{\pi_k} I_{[k \in s]}$.

Assume now that the population U is split into H sub-populations U_h called strata. Let $N_h = |U_h|$, $h \in \{1, ..., H\}$ be the auxiliary information to be taken into account. We split the sample s into H sub-samples s_h defined by $s_h = s \cap U_h$. Let $n_h(s) = |s_h|$ be the size of the sub-sample s_h.

Ideally, to use the auxiliary information at the sampling stage would be best. Here, a simple random stratified sampling (SRS stratified) with a proportional allocation $N_h n/N$ would be more efficient than a SRS. For such a SRS stratified, the set of admissible samples is $\mathcal{S}_{SRSstratified} = \{s \in \mathcal{S}, \forall h \in [1, H], n_h(s) = N_h n/N\}$, and the sampling design is $p(s) = \prod_{h \in [1,H]} \frac{1}{\binom{N_h}{n_h}}, \quad s \in \mathcal{S}_{SRSstratified}$. Once again, our point is precisely to consider setting where the auxiliary information becomes available *posterior* to this sampling stage

3.2 Conditional Inference

The *a posteriori* stratification with an initial SRS was described by Rao(1985) and Tillé(1998). A sample s_0 of size $n(s_0) = n$ is selected. We observe the sizes of the strata sub-samples: $n_h(s_0) = \sum_{k \in U_h} I_{[k \in s]}$, $h \in [1, H]$. We assume that $\forall h, n_h(s_0) > 0$. We then consider the event:

$$A_0 = \{s \in \mathcal{S}, \forall h \in [1, H], n_h(s) = n_h(s_0)\}.$$

It is clear that $s_0 \in A_0$, so A_0 is not empty.

We consider now the conditional probability: $p^{A_0}(.) = p(./A_0)$ which will be used as far inference is concerned. The conditional inclusion probabilities are denoted

$$\pi_k^{A_0} = p^{A_0}\left([I_{[k\in s]} = 1]\right) = \mathbb{E}^{A_0}\left(I_{[k\in s]}\right) = p\left([I_{[k\in s]} = 1] \cap A_0\right)/p(A_0).$$

Accordingly, we define the conditional sampling weights: $d_k^{A_0} = \frac{1}{\pi_k^{A_0}}$.

Proposition 1. *1. The conditional probability p^{A_0} is the law of a stratified simple random sampling with allocation $(n_1(s_0), ..., n_H(s_0))$,*

2. For a unit k of the strata h: $\pi_k^{A_0} = \dfrac{n_h(s_0)}{N_h}$ and $d_k^{A_0} = \dfrac{N_h}{n_h(s_0)}$.

Proof. $|A_0| = \binom{N_1}{n_1(s_0)} \times ... \times \binom{N_H}{n_H(s_0)}$.
$\forall s \in A_0$, $p^{A_0}(s) = 1/|A_0|$. So we have:

$$\begin{aligned}
p^{A_0}(s) &= I_{[s\in A_0]} \frac{1}{\prod_{h\in[1,H]} \binom{N_h}{n_h(s_0)}} \\
&= I_{[s\in A_0]} * \prod_{h\in[1,H]} \frac{1}{\binom{N_h}{n_h(s_0)}} \\
&= \prod_{h\in[1,H]} I_{[n_h(s)=n_h(s_0)]} * \frac{1}{\binom{N_h}{n_h(s_0)}}
\end{aligned}$$

and we recognize the probability law of a stratified simple random sampling with allocation $(n_1(s_0), ..., n_H(s_0))$.
2. follows immediately. □

Note that

$$\mathbb{E}^{A_0}\left(\sum_{k\in U} \frac{y_k}{\pi_k} I_{[k\in s]}\right) = \sum_{k\in U} \frac{y_k}{\pi_k}\pi_k^{A_0} = \sum_h \sum_{k\in U_h} y_k \frac{N n_h(s_0)}{n N_h},$$

so that the genuine HT estimator is conditionally biased in this framework.
Even if, as Tillé(1998) mentioned, it is possible to correct this bias simply by retrieving it from the HT estimator, it seems more coherent to use another linear estimator constructed like the HT estimator but, this time, using the conditional inclusion probabilities.

Remark that in practice A_0 should not be too small. The idea is that for any unit k, we should be able to find a sample s such that $s \in A_0$ and $k \in s$. Thus, all the units of U have a positive conditional inclusion probability.

Definition 1. *The **conditional HT estimator** is defined as:*

$$\hat{t}_{y,CHT} = \sum_{k\in U} \frac{y_k}{\pi_k^{A_0}} I_{[k\in s]}$$

The conditional Horvitz-Thompson (CHT) etimator is obviously conditionally unbiased and, therefore, unconditionally unbiased.

This estimator is in fact the classical post-stratification estimator obtained from a model-assisted approach (see Särndal et Al.(1992) for example). However, conditional inference leads to a different derivation of the variance, which appears to be more reliable as we will see in next subsection.

3.3 Simulations

In this part, we will compare the punctual estimations of a total according to two strategies: (SRS design + conditional (post-stratification) estimator) and (SRS design + HT estimator).

The population size is $N = 500$, the variable y is a quantitative variable drawn

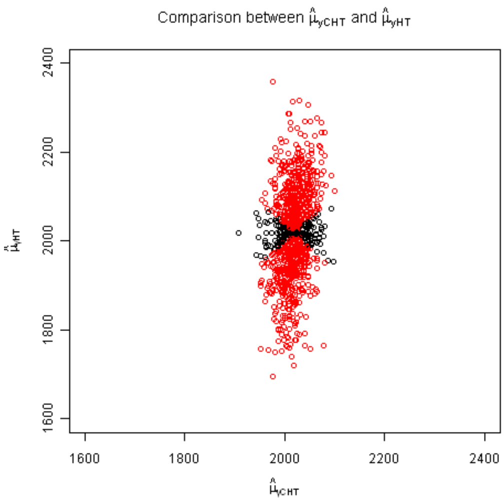


Figure 1: Punctual Estimation

from a uniform distribution over the interval $[0, 4000]$. The population is divided into 4 strata corresponding to the values of y_k (if $y_k \in [0, 1000[$ then k belongs to the strata 1 and so on ...). The auxiliary information will be the size of each strata in the population. In this example, we get $N_1 = 123$, $N_2 = 123$, $N_3 = 132$ and $N_4 = 122$.

The finite population stays fixed and we simulate with the software R $K = 10^3$ simple random samples of size $n = 100$. Two estimators of the mean $\mu_y = \frac{1}{N}\sum_{k\in U} y_k$ are computed and compared. The first one is the HT estimator: $\hat{\mu}_{y,HT} = \frac{1}{n}\sum_{k\in s} y_k$ and the second one is the conditional estimator: $\hat{\mu}_{y,CHT} = \frac{1}{N}\sum_h \sum_{k\in U_h} y_k \frac{N_h}{n_h(s)} I_{[k\in s]}$.

On Figure 1, we can see the values of $\hat{\mu}_{y,HT}$ and $\hat{\mu}_{y,CHT}$ for each of the 10^3 simulations. The red dots are those for which the conditional estimation is closer to the true value $\mu_y = 2019.01$ than the unconditional estimation; red dots represents

83.5% of the simulations. Moreover, the empirical variance of the conditional estimator is clearly smaller than the empirical variance of the unconditional estimator.

This is completely coherent with the results obtained for the post-stratification estimator in an model-assisted approach (see Särndal et Al.(1992) for example). However, what is new and fundamental in the conditional approach, is to understand that for one fixed sample, the conditional bias and variance are much more reliable than the unconditional bias and variance. The theoretical study of the conditional variance estimation is a subject still to be developed.

3.4 Discussion

1. The traditional sampling strategy is defined as a couple (sampling design + estimator). We propose to define here the strategy as a triplet (sampling design + conditional sampling probability + estimator).

2. We have conditioned on the event: $A_0 = \{s \in \mathcal{S}, \forall h \in [1, H]\ n_h(s) = n_h(s_0)\}$. Under a SRS, it is similar to use the HT estimators of the sizes of the strata in the conditioning, that is to use $\mathbf{\Phi}(s) = (\hat{N}_1(s), ..., \hat{N}_H(s))^t$, where $\hat{N}_h(s) = \sum_{k\in U_h} \frac{I_{[k\in s]}}{\pi_k} = \frac{N}{n} n_h(s)$. Then, $A_0 = \{s \in \mathcal{S}, \mathbf{\Phi}(s) = \mathbf{\Phi}(s_0)\}$. We will see in Section 5 the importance of this remark.

3. The CHT estimations of the sizes of the strata are equal to the true strata sizes N_h, which means that the CHT estimations, in this setting, have the calibration property for the auxiliary information of the size of the strata. Hence, conditional inference gives a theoretical framework for the current practice of calibration on auxiliary variables.

4 A Posteriori Conditional Poisson Stratification

Rao(1985), Tillé(1999) and Andersson (2005) mentioned that a posteriori stratification in a more complex setting than an an initial SRS is not a trivial task, and that one must rely on approximate procedures. In this section, we show that it is possible to determine the conditional sampling design and to compute exactly the conditional inclusion probabilities for an a posteriori stratification with a conditional Poisson sampling of size n.

4.1 Conditional Inference

Let $\tilde{p}(s) = \prod_{k\in s} p_k \prod_{k\in\bar{s}}(1 - p_k)$ be a Poisson sampling with inclusion probabilities $\mathbf{p} = (p_1, \ldots, p_N)^t$, where $p_k \in]0, 1]$ and $\bar{s}$ is the complement of s in U. Under a Poisson sampling, the units are selected independently.

By means of rejective technics, a conditional Poisson sampling of size n can be implemented from the Poisson sampling. Then, the sampling design is:

$$p(s) = K^{-1}\mathbf{1}_{|s|=n}\prod_{k\in s} p_k \prod_{k\in\bar{s}}(1-p_k),$$

where $K = \sum_{s,|s|=n}\prod_{k\in s} p_k \prod_{k\in\bar{s}}(1-p_k)$.
The inclusion probabilities $\pi_k = f_k(U,\mathbf{p},n)$ may be computed by means of a recursive method:

$$f_k(U,\mathbf{p},n) = \frac{p_k}{1-p_k}\frac{n}{\sum_{l\in U}\frac{p_l}{1-p_l}(1-f_l(U,\mathbf{p},n-1))}(1-f_k(U,\mathbf{p},n-1))$$

where $f_k(U,\mathbf{p},0)=0$.
This fact was proven by Chen et al.(1994) and one can also see Deville (2000), Matei and Tillé (2005), and Bondesson(2010). An alternative proof is given in Annex 1.
It is possible that the initial π_k of the conditional Poisson sampling design are known instead of the p_k's. Chen et al.(1994) have shown that it is possible to inverse the functions $f_k(U,\mathbf{p},n)$ by the means of an algorithm which is an application of the Newton method. One can see also Deville (2000) who gave an enhanced algorithm.

Assume that a posteriori, thanks to some auxiliary information, the population is stratified in H strata U_h, $h\in[1,H]$. The size of the strata U_h is known to be equal to N_h, and the size of the sub-sample s_h into U_h is $n_h(s_0)>0$. We consider the event $A_0=\{s\in\mathcal{S},\forall h\in[1,H],n_h(s)=n_h(s_0)\}$.

Proposition 2. *With an initial conditional Poisson sampling of size n:*

1. *The probability conditional to the sub-samples sizes of the "a posteriori strata", $p^{A_0}(s)=p(s/A_0)$, is the probability law of a* ***stratified sampling*** *with (independent)* ***conditional Poisson sampling of size*** $n_h(s_o)$ ***in each stratum****,*
2. *The conditional inclusion probability $\pi_k^{A_0}$ of an element k of the strata U_h is the inclusion probability of a conditional Poisson sampling of size $n_h(s_o)$ in a population of size N_h.*

Proof. 1. For a conditional Poisson of fixe size n, a vector $(p_1,\ldots,p_N)^t$ exists, where $p_k\in]0,1]$, such that:

$$p(s) = K^{-1}\mathbf{1}_{|s|=n}\prod_{k\in s} p_k \prod_{k\in\bar{s}}(1-p_k),$$

where $K = \sum_{s,|s|=n}\prod_{k\in s} p_k \prod_{k\in\bar{s}}(1-p_k)$.
We remind that $A_0=\{s\in\mathcal{S},\forall h\in[1,H],n_h(s)=n_h(s_0)\}$
Then:

$$\begin{aligned} p(A_0) &= K^{-1}\tilde{p}\left(\bigcap_{h\in[1,H]}[n_h(s)=n_h(s_0)]\right) \\ &= K^{-1}\prod_{h\in[1,H]}\tilde{p}\left([n_h(s)=n_h(s_0)],\right) \end{aligned}$$

where, $\tilde{p}(.)$ is the law of the original Poisson sampling. Let $s \in A_0$, then:

$$\begin{aligned} p^{A_0}(s) &= \frac{p(s)}{p(A_0)} \\ &= \frac{K^{-1}\prod_{h=1,\ldots,H}\left[\prod_{k\in s_h} p_k \prod_{k\in \bar{s}_h}(1-p_k)\right]}{K^{-1}\prod_{h\in[1,H]}\tilde{p}([n_h(s)=n_h(s_0)])} \\ &= \prod_{h=1,\ldots,H}\frac{\prod_{k\in s_h} p_k \prod_{k\in \bar{s}_h}(1-p_k)}{\tilde{p}([n_h(s)=n_h(s_0)])} \\ &= \prod_{h=1,\ldots,H}\frac{\prod_{k\in s_h} p_k \prod_{k\in \bar{s}_h}(1-p_k)}{\sum_{s_h,|s_h|=n_h(s_0)}\prod_{k\in s-h} p_k \prod_{k\in \bar{s}_h}(1-p_k)}, \end{aligned}$$

which is the sampling design of a stratified sampling with independent conditional Poisson sampling of size $n_h(s_o)$ in each stratum.
2. follows immediately. □

Definition 2. *In the context of conditional inference on the sub-sample sizes of posteriori strata, under an initial conditional Poisson sampling of size n, the* ***conditional HT estimator*** *of the total t_y is:*

$$\hat{t}_{y,CHT} = \sum_{k\in s}\frac{y_k}{\pi_k^{A_0}}.$$

The conditional variance can be estimated by means of one of the approximated variance formulae developed for the conditional Poisson sampling of size n. See for example Matei and Tillé(2005), or Andersson(2004).

4.2 Simulations

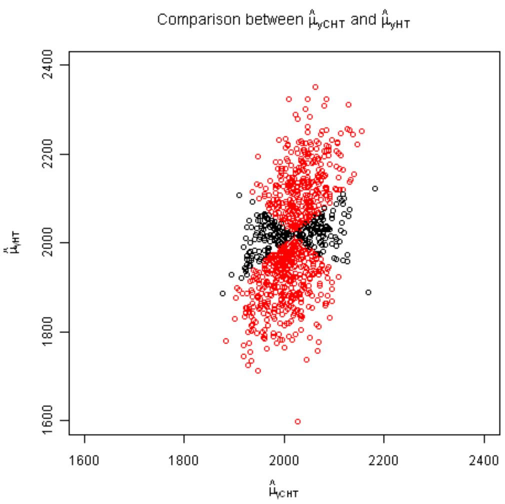


Figure 2: Punctual Estimation

We take the same population as in subsection 3.3. The sampling design is now a conditional Poisson sampling of size $n = 100$. The probabilities p_k of the underlying Poisson design have been generated randomly, in order that $\sum_{k \in U} p_k = n$ and $p_k \in [0.13; 0.27]$.
$K = 10^3$ simulations were performed. Figure 2 shows that the punctual estimation of the mean of y is globally better for conditional inference. According to 77.3% of the simulations the conditional estimator is better than the unconditional estimator (red dots). The empirical variance as well is clearly better for the conditional estimator.

4.3 Discussion

This method allows to compute exact conditional inclusion probabilities in an "a posteriori stratification" under conditional Poisson of size n. However, one can figure out that this method can be used for any unequal probabilities sampling design, had the sampling frame been randomly sorted.

5 Conditioning on the Horwitz-Thompson estimator of an auxiliary variable

In the previous sections, we used the sub-sample sizes in the strata $n_h(s)$ to condition on. The good performances of this conditional approach result from the fact that the sizes of the sub-sample are important characteristics of the sample that are often used at the sampling stage. So, it was not surprising that the use of this information at the estimation stage would enhance the conditional estimators.

Another statistic that characterizes the representativeness of a sample is its HT estimator of the mean μ_x (or total t_x) of an auxiliary variable. This statistic is used at the sampling stage in balanced sampling for example. So, as the sub-sample sizes into the strata, this statistic should produce good results in a conditional approach restraining the inference to the samples for which the HT estimation of μ_x are equal to the value $\hat{\mu}_0 = \hat{\mu}_{x,HT}(s_0)$ of the selected sample s_0 .

In fact, we want the (conditional) set of the possible samples to be large enough in order that all conditional inclusion probabilities be different from zero. It is therefore convenient to consider the set of samples that give HT estimations not necessarily strictly equal to $\hat{\mu}_0$ but close to $\hat{\mu}_0$. Let $\varphi = [\hat{\mu}_0 - \varepsilon, \hat{\mu}_0 + \varepsilon]$, for some $\varepsilon > 0$.

The set A_φ of possible samples in our conditional approach will be:

$$A_\varphi = \{s \in \mathcal{S},\ \hat{\mu}_{x,HT}(s) \in [\hat{\mu}_0 - \varepsilon, \hat{\mu}_0 + \varepsilon]\}.$$

The conditional inclusion probability of a unit k is:

$$\begin{aligned} \pi_k^{A_\varphi} &= p\left([k \in s] /\ [\hat{\mu}_{x,HT}(s) \in [\hat{\mu}_0 - \varepsilon, \hat{\mu}_0 + \varepsilon]]\right) \\ &= \frac{p\left(\{s \in \mathcal{S},\ k \in s \text{ and } \hat{\mu}_{x,HT}(s) \in [\hat{\mu}_0 - \varepsilon, \hat{\mu}_0 + \varepsilon]\}\right)}{p(A_\varphi)}. \end{aligned}$$

If $\hat{\mu}_0 = \mu_X$ then we are in a good configuration, because we are in a balanced sampling situation and the $\pi_k^{A_\varphi}$ will certainly stay close to the π_k.
If $\widehat{\mu}_0 \gg \mu_X$ say, then the sample s_0 is unbalanced, which means that in average, its units have a too large contribution x_k/π_k, either because they are too big (x_k large) or too heavy ($d_k = \frac{1}{\pi_k}$ too large). In this case, the samples in A_φ are also ill-balanced, because balanced on $\widehat{\mu}_0$ instead of μ_X: $\sum_{k\in s} \frac{x_k}{\pi_k} \approx \widehat{\mu}_0$. But conditioning on this information will improve the estimation. Indeed, the $\pi_k^{A_\varphi}$ will be different from the π_k. For example, a unit k with a big contribution ($\frac{x_k}{\pi_k}$ large) has more chance to be in a sample of A_φ than a unit l with a small contribution. So, we can expect that $\pi_k^{A_\varphi} > \pi_k$ and $\pi_l^{A_\varphi} < \pi_l$. And, in consequence, the conditional weight d_k^φ will be lower than d_k and d_l^φ higher than d_l, which will "balance" the samples of A_φ.

Discussion:

- we can use different ways in order to define the subset φ. One way is to use the distribution function of $\Phi(s)$, denoted $G(u)$ and to define φ as a symmetric interval:

  $$\varphi = \left[G^{-1}(\max\{G(\Phi(s_0)) - \frac{\alpha}{2}, 0\}), G^{-1}(\min\{G(\Phi(s_0)) + \frac{\alpha}{2}, 1)\}\right],$$

  where $\alpha = 5\%$ for example.

  Hence,

  $$A_\varphi = \{s \in \mathcal{S}, \Phi(s) \in \left[G^{-1}(\max\{G(\Phi(s_0)) - \frac{\alpha}{2}, 0\}), G^{-1}(\min\{G(\Phi(s_0)) + \frac{\alpha}{2}, 1)\}\right]\},$$

  and $p(A_\varphi) \leq \alpha$.

  As the *cdf* $G(u)$ is unknown in general, one has to replace it by an estimated *cdf* of $\Phi(s)$, denoted $\hat{G}_K(u)$, computed by means of simulations.

6 Generalization: Conditional Inference Based on Monte Carlo simulations.

In this section, we consider a general initial sample design $p(s)$ with the inclusion probabilities π_k. We condition on the event $A_\varphi = \mathbf{\Phi}^{-1}(\varphi) = \{s \in \mathcal{S}, \mathbf{\Phi}(s) \in \varphi\}$. For example, we can use $\Phi(s) = \sum_{k\in s} \frac{\mathbf{x}_k}{\pi_k}$ the unconditional HT estimator of t_x and $\varphi = [\varphi_1, \varphi_2]$ an interval that contains $\Phi(s_0) = \sum_{k\in s_0} \frac{\mathbf{x}_k}{\pi_k}$, the HT estimation of t_x with the selected sample s_0. In other words, we will take into account the information that the HT estimator of the total of the auxiliary variable x lies in some region φ.

The mathematical expression of $\pi_k^{A_\varphi}$ is straightforward:

$$\pi_k^{A_\varphi} = p([k \in s]/A_\varphi) = \frac{\sum_s p(s)\mathbf{1}_{s\in A_\varphi}\mathbf{1}_{[k\in s]}}{p(A_\varphi)}.$$

But effective computation of the $\pi_k^{A_\varphi}$'s may be not trivial if the distribution of $\mathbf{\Phi}$ is complex. Tillé(1998) used an asymptotical approach to solve this problem when $\Phi(s) = \sum_{k\in s} \frac{x_k}{\pi_k}\mathbf{1}_{[k\in s]}$; he has used normal approximations for the conditional and unconditional laws of Φ.
In the previous sections, we have given examples where we were able to compute the $\pi_k^{A_\varphi}$'s (and actually the $p^{A_\varphi}(s)$'s) exactly. In this section, we give a general Monte Carlo method to compute the $\pi_k^{A_\varphi}$.

6.1 Monte Carlo

We will use Monte Carlo simulations to estimate $\mathbb{E}(\mathbf{1}_{A_\varphi}\mathbf{1}_{[k\in s]})$ and $\mathbb{E}(\mathbf{1}_{A_\varphi})$. We repeat independently K times the sample selection with the sampling design $p(s)$, thus obtaining a set of samples $(s_1, \ldots, s_K)$. For each simulation i, we compute $\mathbf{\Phi}(s_i)$ and $I_{A_\varphi}(s_i)$. Then we compute $N+1$ statistics:

$$\begin{aligned} M^{A_\varphi} &= \sum_{i=1}^{K} \mathbf{1}_{A_\varphi}(s_i) \\ \forall k \in U, M_k^{\varphi} &= \sum_{i=1}^{K} \mathbf{1}_{A_\varphi}(s_i)\mathbf{1}_{[k\in s_i]} \end{aligned}$$

We obtain a consistent estimator of $\pi_k^{A_\varphi}$, as $K \to +\infty$:

$$\hat{\pi}_k^{A_\varphi} = \frac{M_k^{\varphi}/K}{M^{A_\varphi}/K} = \frac{M_k^{\varphi}}{M^{A_\varphi}} \tag{1}$$

6.2 Point and variance estimations in conditional inference

Definition 3. *The Monte Carlo estimator of the total t_y is the conditional Horvitz-Thompson estimator of t_y after replacing the conditional inclusion probabilities by their Monte Carlo approximations:*

$$\hat{t}_{y,MC} = \sum_{k\in s_0} \frac{1}{\hat{\pi}_k^{A_\varphi}} y_k$$

The Monte Carlo estimator of the variance of $\hat{t}_{y,MC}$ is:

$$\widehat{\mathbb{V}}(\hat{t}_{y,MC}) = \sum_{k,l\in s_0} \frac{1}{\hat{\pi}_{k,l}^{A_\varphi}} \frac{y_k}{\hat{\pi}_k^{A_\varphi}} \frac{y_l}{\hat{\pi}_l^{A_\varphi}} (\hat{\pi}_{k,l}^{A_\varphi} - \hat{\pi}_k^{A_\varphi}\hat{\pi}_l^{A_\varphi}),$$

where

$$\hat{\pi}_{k,l}^{A_\varphi} = \frac{\sum_{i=1}^{K} t_y \mathbf{1}_{A_\varphi}(s_i)\mathbf{1}_{[k\in s_i]}\mathbf{1}_{[l\in s_i]}}{\sum_{i=1}^{K} \mathbf{1}_{A_\varphi}(s_i)}.$$

Fattorini(2006) established that $\hat{t}_{y,MC}$ is asymptotically unbiased as $M^{A_\varphi} \to \infty$, and that its mean squared error converges to the variance of $\hat{t}_{y,HT}$.

Thompson and Wu (2008) studied the rate of convergence of the estimators $\hat{\pi}_k^{A_\varphi}$ and of the estimator $\hat{t}_{y,MC}$ following Chebychev's inequality. Using normal approximation instead of the Chebychev's inequality gives more precise confidence intervals. We have thus a new confidence interval for $\hat{\pi}_k^{A_\varphi}$:

$$p\left(|\hat{\pi}_k^{A_\varphi} - \pi_k^{A_\varphi}| < F^{-1}((1-\alpha)/2)\sqrt{\frac{1}{4M^{A_\varphi}}}\right) \leq \alpha,$$

where F is the distribution function of the normal law $\mathcal{N}(0,1)$.
As for the relative bias, standard computation leads to:

$$\begin{aligned} p\left(\frac{|\hat{t}_{y,CHT} - \tilde{t}_{y,CHT}|}{\hat{t}_{y,CHT}} \leq \varepsilon\right) &\geq 1 - 4 \times \sum_{k\in s}\left[1 - F\left(\frac{\epsilon}{1+\epsilon}\sqrt{M^{A_\varphi}\pi_k^{A_\varphi}}\right)\right] \\ &\geq 1 - 4n\frac{1}{\sqrt{2\pi}}\frac{1+\epsilon}{\sqrt{M^{A_\varphi}\epsilon^2\pi_0}} \cdot e^{-\left(\frac{M^{A_\varphi}\epsilon^2}{(1+\epsilon)^2}\pi_0\right)}, \qquad (2) \end{aligned}$$

where $\pi_0 = \min\{\pi_k^{A_\varphi}, k \in U\}$. We used the inequality $1 - F(u) \leq \frac{1}{\sqrt{2\pi}}\frac{e^{-u^2}}{u}$ which is verified for large u.

The number K of simulations is set so that $\sum_{i=1}^{K} I_{A_\varphi}(s_i)$ reaches a pre-established M^{A_φ} value. Because of our conditional framework, K is a stochastic variable which follows a negative binomial distribution and we have $E(K) = \dfrac{M^{A_\varphi}}{p(A_\varphi)}$. For instance, if $p(A_\varphi) = 0.05 = 5\%$, with $M^{A_\varphi} = 10^6$, we expect $E(K) = 2.10^7$ simulations.

7 Conditional Inference Based on Monte Carlo Method in Order to Adjust for Outlier and Strata Jumper

We will apply the above ideas to two examples close to situations that can be found in establishments surveys: outlier and strata jumper.
We consider an establishments survey, performed in year "*n+1*", and addressing year "*n*". The auxiliary information x which is the turnover of the year "*n*" is not known at the sampling stage but is known at the estimation stage (this information may come from, say, the fiscal administration).

7.1 Outlier

In this section, the auxiliary variable x is simulated following a gaussian law, more precisely $x_k \sim \mathfrak{N}(8\ 000, (2\ 000)^2)$ excepted for unit $k = 1$ for which we assume that

$x_1 = 50\ 000$. The unit $k = 1$ is an outlier. The variable of interest y is simulated by the linear model

$$y_k \;=\; 1000 \;+\; 0.2\ x_k \;+\; u_k,$$

where $u_k \sim \mathfrak{N}(0, (500)^2)$, u_k is independent from x_k. The outcomes are $\mu_x = 8\ 531$ and $\mu_y = 2\ 695$.

We assume that the sampling design of the establishments survey is a SRS of size $n = 20$ out of the population U of size $N = 100$ and that the selected sample s_0 contains the unit $k = 1$. For this example, we have repeated the sample selection until the unit 1 has been selected in s_0.
We obtain $\Phi(s_0) = \hat{\mu}_{x,HT}(s_0) = 9\ 970$, which is 17% over the true value $\mu_x = 8\ 531$ and $\hat{\mu}_{y,HT}(s_0) = 3\ 039$ (recall that the true value of μ_y is 2 695).
We set Φ and φ as in section 5 and we use Monte Carlo simulations in order to compute the conditional inclusion probabilities $\hat{\pi}_k^{A_\varphi}$. Each simulation is a selection of a sample following a SRS of size $n = 20$from the fixed population U. Recall that the value of x_k will eventually be known for any unit $k \in U$.

Actually, we use two sets of simulations. The first set is performed in order to estimate the *cdf* of the statistic $\Phi(s) = \hat{\mu}_{x,HT}(s)$ which will be used to condition on. This estimated *cdf* will enable us to construct the interval φ. More precisely, we choose the interval $\varphi = [9\ 793, 10\ 110]$ by the means of the estimated *cdf* of $\Phi(s) = \hat{\mu}_{y,HT}(s)$ and so that $p\left([\hat{\mu}_{x,HT}(s) \in [9\ 793, \hat{\mu}_{x,HT}(s_0)]]\right) = \frac{\alpha}{2} = 2.5\% = p\left([\hat{\mu}_{x,HT}(s) \in [\hat{\mu}_{x,HT}(s_0), 10\ 110]]\right)$.
A_φ is then the set of the possible samples in our conditional approach:

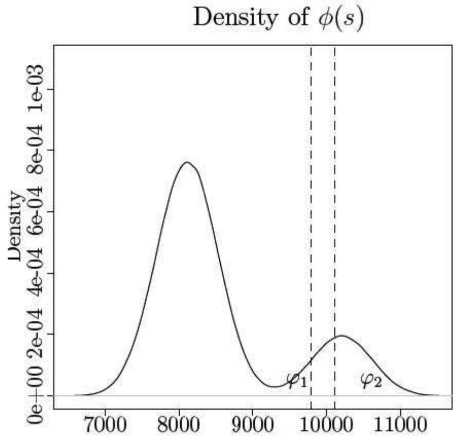
$$A_\varphi = \{s \in \mathcal{S},\ \hat{\mu}_{x,HT}(s) \in [9\ 793, 10\ 110]\}.$$

Note that $p\left([\hat{\mu}_{x,HT}(s) \in [9\ 793, 10\ 110]]\right) = \alpha = 5\%$. A_φ typically contains samples that over-estimate the mean of x.

The second set of Monte Carlo simulations consists in $K = 10^6$ sample selections with a SRS of size $n = 20$ performed in order to estimate the conditional inclusion probabilities $\hat{\pi}_k^{A_\varphi}$. 49 782 (4.98%) simulated samples fall in A_φ, and among them, 49 767 samples contain the outlier, which correspond to the estimated conditional inclusion probability of the outlier: $\hat{\pi}_1^{A_\varphi} = 0.9997$. It means that almost all the samples of A_φ contain the outlier that is mainly responsible for the over-estimation because of its large value of the variable x!
The weight of the unit 1 has changed a lot, it has decreased from $d_k = \frac{1}{0.2} = 5$ to $\hat{d}_k^{A_\varphi} = 1.0003$. The conditional sampling weights of the other units of s_0 are more comparable to their initial weights $d_k = 5$ (see Figure 7.1).

The conditional MC estimator $\hat{\mu}_{y,MC}(s) = \frac{1}{N}\sum_{k\in s} \frac{y_k}{\hat{\pi}_1^{A_\varphi}}$ leads to a much better estimation of μ_y: $\hat{\mu}_{y,MC}(s_0) = 2\ 671$.

Figure 7.1 gives an idea of the conditional inclusion probabilities for all the units of U. Moreover, this graph shows that the correction of the sampling weights $\frac{\hat{d}_k^{A_\varphi}}{d_k} = \frac{\pi_k}{\hat{\pi}_k^{A_\varphi}}$ is



not a monotonic function of x_k, which is in big contrast with calibration techniques which only uses monotonic functions for weight correction purposes.

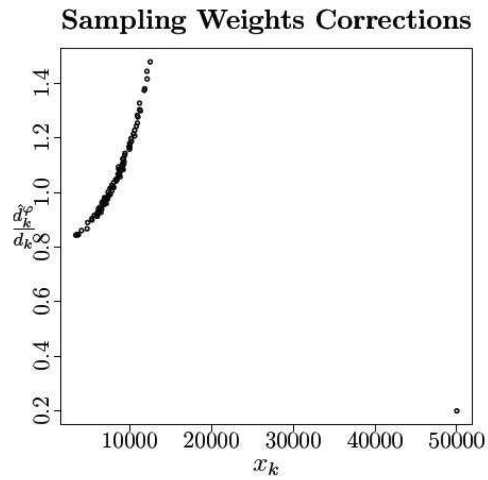


Figure 3: Outlier, Density of $\Phi(s) = \hat{\mu}_{x,HT}(s)$

A last remark concerns the distribution of the statistics $\Phi(s) = \hat{\mu}_{x,HT}(s)$. Figure 3 shows an unconditional distribution with 2 modes and far from gaussian. This shows that in presence of outlier, we can not use the method of Tillé (1999), which assumes a normal distribution for $\hat{\mu}_{x,HT}(s)$.

7.2 Strata Jumper

In this section, the population U is divided into 2 sub-populations: the small firms and the large firms. Let us say that the size is appreciated thanks to the turnover of the firm. Official statistics have to be disseminated for this 2 different sub-populations. Hence, the survey statistician has to split the population into 2 strata

corresponding to the sub-populations. This may not be an easy job because the size of firms can evolve from one year to another.
Here we assume that, at the time when the sample is selected, the statistician does not know yet the auxiliary information x of the turnover of the firm for the year "n", more precisely the strata the firm belongs to for the year "n". Let us assume that he only knows this information for the previous year,"*n-1*". This information is denoted by z. In practice, small firms are very numerous and the sampling rate for this strata is chosen low. On the contrary, large firms are less numerous and their sampling rate is high.

When a unit is selected among the small firms but eventually happens to be a large unit of year "n", we call it a strata jumper. At the estimation stage, when the information x becomes available, this unit will obviously be transferred to strata 2 . This will bring a problem, not due to its y-value (which may well be typical in strata 2) but to its sampling weight, computed according to strata 1 (the small firms), and which will appear to be very large in comparison to the other units in strata 2 at the estimation stage.

In our simulations, the population U is split in 2 strata, by means of the auxiliary variable z: U_1^z, of size $N_1^z = 10\ 000$, is the strata of presumed small firms and U_2^z, of size $N_2^z = 100$, the strata of presumed large firms.
The auxiliary variable x, which is the turnover of the year "n" known after collection, is simulated under a gaussian law $\mathfrak{N}(8\ 000, (2\ 000)^2)$ for the units of the strata U_2^z and for one selected unit of the strata U_1^z. Let us say that this unit, the strata jumper, is unit 1.
Our simulation gives $x_1 = 8\ 002$. The variable of interest y is simulated by the linear model $y_k = 1000 + 0.2\ x_k + u_k$, where $u_k \sim \mathfrak{N}(0, (500)^2)$, u_k and x_k independent. We do not simulate the value of x and y for the other units of the strata U_1^z because we will focus on the estimation of the mean of y for the sub-population of large firms of year n U_2^x: $\mu_{y,2} = \frac{1}{N_2} \sum_{k \in U_2^x} y_k$. We find $\mu_{x,2} = 8\ 138$ and $\mu_{y,2}$ is 2 606.

The sampling design of the establishments survey is a stratified SRS of size $n_1 = 400$ in U_1^z and $n_2 = 20$ in U_2^z. We assume that the selected sample s_0 contains the unit $k = 1$. In practice, we repeat the sample selection until the unit 1 (the strata jumper) has been selected.

As previously, Φ and φ are defined as in Section 5.
We use Monte Carlo simulations in order to compute the conditional inclusion probabilities $\hat{\pi}_k^{A_\varphi}$. A simulation is a selection of a sample with stratified SRS of size $n_1 = 400$ in U_1^z and $n_2 = 20$ in U_2^z.

We choose the statistic $\Phi(s) = \hat{\mu}_{x,2,HT}(s)$ in order to condition on. $K = 10^6$ simulations are performed in order to estimate the *cdf* of $\Phi(s)$ and the conditional inclusion probabilities.
Our simulations give $\Phi(s_0) = \hat{\mu}_{x,2,HT}(s_0) = 9\ 510$, which is far from the true value $\mu_{x,2} = 8\ 138$ and $\hat{\mu}_{y,2,HT}(s_0) = 3\ 357$ (recall that the true value of $\mu_{y,2}$ is 2 606).

We choose the interval $\varphi = [8\ 961, 10\ 342]$ by the means of the estimated *cdf* of $\Phi(s) = \hat{\mu}_{x,2,HT}(s)$ and so that $p\left([\hat{\mu}_{x,2,HT}(s) \in [8\ 961, 10\ 342]]\right) = \alpha = 5\%$.
A_φ is then the set of the possible samples in our conditional approach:

$$A_\varphi = \{s \in \mathcal{S},\ \hat{\mu}_{x,HT}(s) \in [8\ 961, 10\ 342]\}.$$

All samples in A_φ over-estimate the mean of x.

Among the 10^6 simulations, 49 778 simulated samples (4.98%) belongs to A_φ. 55% of them contains the strata jumper, which gives the estimated conditional inclusion probability of the strata jumper $\hat{\pi}_1^{A_\varphi} = 0.55$. It is not a surprise that the strata jumper is in one sample of A_φ over two. Indeed, its initial sampling weight $d_1 = \frac{10\ 000}{400} = 25$ is high in comparison to the weights $d_k = \frac{100}{20} = 5$ of the other selected units of the strata U_2^x, and its contribution $\frac{25x_1}{N_2}$ contributes to over-estimate the mean of x .

The conditional inclusion probabilities for the other units of U_2^x are comparable to their initial $\pi_k = 0.2$ (see Figure 4).

The conditional MC estimator $\hat{\mu}_{y,2,MC}(s) = \frac{1}{N}\sum_{k\in s}\frac{y_k}{\hat{\pi}_1^{A_\varphi}}$ leads to a better estimation of $\mu_{y,2}$: $\hat{\mu}_{y,2,MC}(s_0) = 2\ 649$.

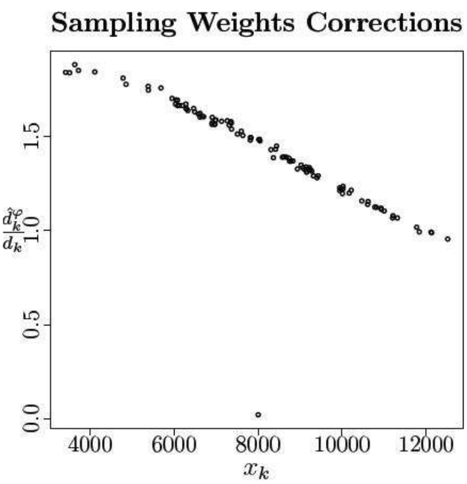


Figure 4: Strata Jumper, Sampling Weight Corrections

Figure 4 shows that sampling weights correction is here a non-monotonic function of the variable x. We point out that the usual calibration method would not be able to perform this kind of weights correction because the calibration function used to correct the weights should be monotonic.
Similarly to the outlier setting, the unconditional distribution of the statistics $\Phi(s) = \hat{\mu}_{x,2,HT}(s)$ has 2 modes and is far from gaussian.

8 Conclusion

At the estimation stage, a new auxiliary information can reveal that the selected sample is imbalanced. We have shown that a conditional inference approach can take into account this information and leads to a more precise estimator than the unconditional Horvitz-Thompson estimator in the sense that the conditional estimator is unbiased (conditionally and unconditionally) and that the conditional variance is more rigorous in order to estimate the precision a posteriori.
In practise, we recommend to use Monte Carlo simulations in order to estimate the conditional inclusion probabilities.
This technic seems particularly adapted to the treatment of outliers and strata-jumpers.

A Annex 1: Inclusion Probability with Conditional Poisson Sampling

Proof. The event $\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]$ is independent of the events $[I_{[k\in s]} = 0]$ and $[I_{[k\in s]} = 1]$ in the Poisson model. So we can write:

$$p\left(\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]\right) = p\left(\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]/[I_{[k\in s]} = 0]\right) \tag{3}$$

$$= p\left(\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]/[I_{[k\in s]} = 1]\right) \tag{4}$$

Equation (3) gives:

$$\begin{aligned} p\left(\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]/[I_{[k\in s]} = 0]\right) &= p\left(\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]/[I_{[k\in s]} = 0]\right) \\ &= p\left(\left[\sum_{l\in U} I_{[l\in s]} = n-1\right]/[I_{[k\in s]} = 0]\right) \\ &= \frac{p\left(\left[\sum_{l\in U} I_{[l\in s]} = n-1\right]\right) p\left([I_{[k\in s]} = 0]/\left[\sum_{l\in U} I_{[l\in s]} = n-1\right]\right)}{p\left([I_{[k\in s]} = 0]\right)} \\ &= \frac{p\left(\left[\sum_{l\in U} I_{[l\in s]} = n-1\right]\right)(1 - f_k(N, \mathbf{p}, n-1))}{1-p_k}, \end{aligned}$$

and equation (4) gives:

$$\begin{aligned} p\left(\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]/[I_{[k\in s]} = 1]\right) &= p\left(\left[\sum_{l\in U, l\neq k} I_{[l\in s]} = n-1\right]/[I_{[k\in s]} = 1]\right) \\ &= p\left(\left[\sum_{l\in U} I_{[l\in s]} = n\right]/[I_{[k\in s]} = 1]\right) \\ &= \frac{p\left(\left[\sum_{l\in U} I_{[l\in s]} = n\right]\right) p\left([I_{[k\in s]} = 1]/\left[\sum_{l\in U} I_{[l\in s]} = n\right]\right)}{p\left([I_{[k\in s]} = 1]\right)} \\ &= \frac{p\left(\left[\sum_{l\in U} I_{[l\in s]} = n\right]\right) f_k(N, \mathbf{p}, n)}{p_k}. \end{aligned}$$

So we have:

$$\begin{aligned} f_k(U, \mathbf{p}, n) &= (1 - f_k(U, \mathbf{p}, n-1))\frac{p_k}{1-p_k}\frac{p\left(\left[\sum_{l\in U} I_{[l\in s]} = n-1\right]\right)}{p\left(\left[\sum_{l\in U} I_{[l\in s]} = n\right]\right)} \\ &= (1 - f_k(U, \mathbf{p}, n-1))\frac{p_k}{1-p_k} h(U, \mathbf{p}, n) \end{aligned}$$

And we can use the property $\sum_{k\in U} f_k(U, \mathbf{p}, n) = \sum_{k\in U} \pi_k = n$ to compute $h(U, \mathbf{p}, n)$ and conclude. □

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